

Deformation data, Belyi maps, and the local lifting problem

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Abstract

We prove existence and nonexistence results for certain differential forms in positive characteristic, called *good deformation data*. Some of these results are obtained by reduction modulo p of Belyi maps. As an application, we solve the local lifting problem for groups with Sylow p -subgroup of order p .

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Introduction

The local lifting problem Let k be an algebraically closed field of characteristic $p > 0$, and let G be a finite group. A *local G -action* is a continuous, faithful and k -linear action $\phi : G \hookrightarrow \text{Aut}_k(k[[x]])$ of G on the ring of formal power series $k[[x]]$. If such an action exists, then $G = P \rtimes C$, where P is a p -group and $C \cong \mathbb{Z}/m\mathbb{Z}$ is a cyclic group of order m , with $(m, p) = 1$.

The *local lifting problem* concerns the following question. Given a local G -action ϕ , does there exist a lift of ϕ to an R -linear action $\phi_R : G \hookrightarrow R[[x]]$, where R is a complete discrete valuation ring of characteristic zero with residue field k ? If such a lift exists, we will say ϕ *lifts to characteristic zero*. See [6] or [5] for a general discussion of the local lifting problem.

In the previous paper [3] we have shown that, for G the dihedral group of order $2p$, every local G -action can be lifted to characteristic zero. The methods used in [3] apply more generally to the case where the Sylow p -subgroup P of G has order p . In this more general situation not every local G -action lifts. It is easy to write down certain necessary conditions ([3], Proposition 1.3). We already suspected that these conditions are also sufficient ([3], Question 1.4) and reduced the proof of sufficiency to the existence of a certain differential form in characteristic p . This differential form is the deformation datum referred to in the title of the present paper.

Our first new result says that the differential form needed to prove sufficiency always exists. As a consequence, the local lifting problem has now been solved for all groups G with a Sylow p -subgroup P of order p .

Good deformation data Let $G = P \rtimes C$, where P is cyclic of order p and C is cyclic of order m and where m is prime to p . Choose a generator τ of P and denote by $\chi : C \rightarrow \mathbb{F}_p^\times$ the character such that $\sigma\tau\sigma^{-1} = \tau^{\chi(\sigma)}$, for $\sigma \in C$.

Let $\phi : G \hookrightarrow \text{Aut}_k(k[[x]])$ be a local G -action. The *conductor* of ϕ is defined as the (positive) integer

$$h := \text{ord}_x \left(\frac{\phi(\tau)(x)}{x} - 1 \right), \quad (1)$$

where $\text{ord}_x(f)$ denotes the order of zero of a power series $f \in k[[x]]$. It is easy to see that h is prime to p . We say that the local G -action ϕ has *large conductor* if $h > p$ and that it has *small conductor* if $h < p$.

The results of the present paper concern mainly the case of small conductor. Hence we suppose, for the rest of this introduction, that $h < p$.

Let $\phi_R : G \hookrightarrow \text{Aut}_R(R[[x]])$ be a lift of ϕ to characteristic zero. By a construction going back to [7] (see also [8] and [3]), we can associate to ϕ_R a differential form $\omega = f(z)dz$ over a rational function field $k(z)$ with the following properties:

- ω is logarithmic, i.e. of the form $\omega = dg/g$, with $g \in k(z)$,
- ω has a single zero of order $h - 1$.
- There is a faithful action $C \hookrightarrow \text{Aut}_k(k(z))$ such that

$$\sigma^*\omega = \chi(\sigma) \cdot \omega,$$

for $\sigma \in C$.

A differential form ω with these properties is called a *good deformation datum* with *conductor* h . (Note that this definition depends on the character χ , which we regard as fixed.)

Conversely, a good deformation datum ω as above essentially determines a lift of ϕ to characteristic zero. In particular, a local G -action ϕ lifts to characteristic zero if and only if there exists a good deformation datum with the same conductor h .

In Section 1 we give necessary conditions for the existence of a good deformation datum with given conductor (Proposition 1.3). Then we show that these conditions are also sufficient, at least for small conductor (Proposition 1.4). In Section 2 we show that these two results give a complete solution to the lifting problem for groups of the form $G = P \rtimes C$, with $|P| = p$.

Belyi morphisms Let us assume, for simplicity, that $m = 1$ in the situation considered above. Let $\omega = f(z)dz$ be a good deformation datum with conductor h . After a change of coordinate, we may assume that the unique zero of ω is located at $z = \infty$. Let $z = z_1, \dots, z = z_r$ be the positions of the poles of ω . Since ω is logarithmic, it has at most simple poles. Therefore, $r = h + 1$. Moreover, the residue

$$a_i := \text{Res}_{z=z_i}(\omega)$$

is an element of \mathbb{F}_p^\times . Note that $\sum_i a_i = 0$ by the residue theorem. We call the tuple $\mathbf{a} = (a_1, \dots, a_r)$ the *type* of ω .

In Sections 3 to 6 we consider the following problem: given a tuple $\mathbf{a} = (a_1, \dots, a_r) \in (\mathbb{F}_p^\times)^r$ with $\sum_i a_i = 0$, does there exist a good deformation datum ω of type \mathbf{a} ? This is a refined version of the problem discussed above, where we only fix the conductor h .

Given $\mathbf{a} = (a_1, \dots, a_r)$ as above, it is easy to write down a system of algebraic equations whose solutions correspond to good deformation data of type \mathbf{a} . So far, most attempts to construct good deformation data consisted in trying to solve these equations. For instance, [3], Theorem 5.6, relies on the mysterious fact that the system of equations corresponding to a particular type \mathbf{a} has a unique solution. The proof of this fact given in [3] is quite complicated and offers no explanation why this type \mathbf{a} works and others don't.

In this paper we follow a different path. Consider a good deformation datum ω of type $\mathbf{a} = (a_1, \dots, a_r)$. Then to every choice of lifts $A_i \in \mathbb{Z}$ of a_i such that $\sum_i A_i = 0$ one can associate an essentially unique rational function $g \in k(z)$ with the following properties:

- $\omega = dg/g$,
- g has exactly r zeroes/poles, of order A_1, \dots, A_r , and
- $g : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ is branched exactly over $0, 1$ and ∞ . The fiber $g^{-1}(1)$ contains a unique ramification point, with ramification index $h = r - 1$.

If, moreover, ω has small conductor $h < p$ then g is at most tamely ramified. It follows from a theorem of Grothendieck that g is the reduction modulo p of a *Belyi map*, i.e. a rational function $f : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ with exactly three critical values $0, 1, \infty$. The ramification structure of f is the same as that of g ; in particular, it is completely determined by the tuple $\mathbf{A} = (A_1, \dots, A_r)$. We say that f is a *Belyi map of type \mathbf{A}* .

Our new method consists in choosing a suitable lift \mathbf{A} of \mathbf{a} and to construct a Belyi map f of type \mathbf{A} . One then obtains a good deformation datum $\omega = dg/g$ from the reduction g of f modulo p . Whether this is possible or not for a particular type \mathbf{a} can be easily decided, thanks to the following lemma.

Lemma *Given $\mathbf{A} = (A_1, \dots, A_r) \in \mathbb{Z}^r$ with $\sum_i A_i = 0$; set*

$$n_{\mathbf{A}} := \sum_i \max(A_i, 0), \quad k_{\mathbf{A}} = \gcd(A_1, \dots, A_r).$$

Then there exists a Belyi map of type \mathbf{A} if and only if

$$k_{\mathbf{A}} \cdot (r - 1) \leq n_{\mathbf{A}}.$$

It seems that this lemma has been discovered independently, in various forms, by several authors. For instance, it is a special case (the *genus zero case*) of a

result of Boccardo [1]. It also follows from Theorem 1 of [10]. In Section 5 we shall give a simple proof of the above lemma which uses *dessins d'enfants*. This proof is very similar to the proof given in [10]. We have decided to keep it in the paper, because our use of dessins may be considered as more geometric and easier to understand than explicit manipulations with permutations, as in [10].

The lemma, combined with Grothendieck's theory of lifting and reduction of tamely ramified coverings, immediately gives necessary and sufficient conditions for the existence of good deformation data of a given type \mathbf{a} , in the case where $m = 1$. See Corollary 4.2 and Theorem 4.5.

One can adapt the method described above to the case $m = 2$ and hence obtain almost complete results as well. This is done in Section 6.

It is not clear how to adapt the method to the case $m > 2$. However, one can still use the case $m = 1$ to obtain nonexistence results for good deformation data with arbitrary m . It seems that all nonexistence results that have been documented in the literature so far can be explained in this way.

1 Good deformation data

Let k be an algebraically closed field of characteristic $p > 0$. We let $K = k(z)$ denote the rational function field over k , which we regard as the function field of the projective line $X = \mathbb{P}_k^1$.

Let $m \geq 1$ be a positive integer which is prime to p . We choose, once and for all, a primitive m th root of unity $\zeta_m \in k^\times$. We denote by $\sigma : X \xrightarrow{\sim} X$ the automorphism of order m such that $\sigma^*z = \zeta_m z$.

Definition 1.1 A *deformation datum* is a differential form $\omega = f(z) dz \in \Omega_{K/k}^1$ with the following properties:

- (a) ω is an eigenvector for σ , i.e.

$$\sigma^*\omega = \zeta_m^c \omega,$$

for some $c \in \mathbb{Z}$, and

- (b) ω is logarithmic, i.e. $\omega = dg/g$, for some $g \in K^\times$.

A deformation datum ω is called *primitive* if the integer c in (a) is prime to m . It is called *good* if it has a unique zero, i.e. there exists a unique point $x \in X$ with $\text{ord}_x \omega > 0$. If this is the case then the integer

$$h := \text{ord}_x \omega + 1$$

is called the *conductor* of ω .

We remark that in our previous papers the definition of the term 'deformation datum' is more general. Deformation data occur naturally in a variety of contexts, see e.g. [4], [2]. Here we focus on applications to the local lifting

problem, as explained in the introduction. To save paper we have chosen a more restricted setup from the start, suitable for the applications we have in mind.

The present paper is concerned with the following problem.

Problem 1.2 For which m and h does there exist a good deformation datum ω of conductor h ?

The following proposition gives necessary conditions on h and m .

Proposition 1.3 *Fix positive integers m and h and assume that there exists a good deformation datum ω of conductor h . Then the following holds.*

- (i) *The conductor h is prime to p .*
- (ii) *ω is primitive if and only if h is prime to m . If this is the case, then*

$$m|(p-1)$$

(and hence $\zeta_m \in \mathbb{F}_p^\times$) and

$$h \equiv -1 \pmod{m}.$$

- (iii) *If ω is not primitive, then $m|h$ and $\sigma^*\omega = \omega$ (i.e. $c \equiv 0 \pmod{m}$) in Definition 1.1 (a)).*

This is a special case of [3], Lemma 3.3.(v). For convenience, we recall the proof.

Proof: Let $x \in X$ be the unique zero of ω . Choose a local coordinate $w \in k(z)$ at x and a function $g \in k(z)$ such that $\omega = dg/g$. By multiplying g with a p th power, if necessary, we can achieve that $g(x) = 1$. Writing g as a power series in w and computing $\omega = dg/g$, one sees that $h = \text{ord}_x \omega + 1 \not\equiv 0 \pmod{p}$. This proves (i).

The statements (ii) and (iii) are trivial for $m = 1$. We may therefore assume $m > 1$. Since the automorphism $\sigma : X \xrightarrow{\sim} X$ has order m , it has exactly two fixed points, namely $z = 0$ and $z = \infty$. The unique zero x of ω is clearly fixed by σ . Replacing the coordinate z by z^{-1} , if necessary, we may assume that x is the point $z = \infty$ and that $w = z^{-1}$. Now $\sigma^*w = \zeta_m^{-1}w$, and Condition (a) of Definition 1.1 implies

$$h = \text{ord}_x(\omega) + 1 \equiv -c \pmod{m}. \quad (2)$$

In particular, ω is primitive if and only if h is prime to m .

The same argument used to prove (2) shows that

$$\text{ord}_{z=0}(\omega) + 1 \equiv c \pmod{m}. \quad (3)$$

But $\text{ord}_{z=0}(\omega)$ is either equal to -1 or to 0 . In the first case, c and h are divisible by m ; this corresponds to Part (iii) of the proposition. In the second

case, $c \equiv 1 \pmod{m}$, $h \equiv -1 \pmod{m}$ and ω is primitive. This corresponds to Part (ii) of the proposition.

It remains to prove that $m|(p-1)$ in the second case. Let $z = z_1$ be a pole of ω and set $a_1 := \text{Res}_{z=z_1}(\omega)$. Set $z_2 := \sigma(z_1) = \zeta_m z_1$. Condition (a) of Definition 1.1, together with the congruence $c \equiv 1 \pmod{m}$, implies that

$$a_2 := \text{Res}_{z=z_2}(\omega) = \zeta_m^{-1} a_1.$$

Since ω is logarithmic, the residues a_1, a_2 actually lie in $\mathbb{F}_p^\times \subset k^\times$ (see Lemma 1.5 below). Therefore, $\zeta_m \in \mathbb{F}_p^\times$, which is equivalent to $m|(p-1)$. This finishes the proof of the proposition. \square

The next result says that the necessary conditions given by Proposition 1.3 are also sufficient, at least if $h < p$. To keep the statement simple, we first deal with the case $m|h$ (the non-primitive case). Here one can immediately write down a good deformation datum of conductor h (see also [8], §3.5):

$$\omega := \frac{h \, dz}{z^{h+1} - z} = \frac{dg}{g}, \quad \text{with } g := (z^h - 1)/z^h. \quad (4)$$

It therefore suffices to consider the primitive case (Part (ii) of Proposition 1.3).

Proposition 1.4 *Assume $m|(p-1)$ and let h be a positive integer with*

$$h < p \quad \text{and} \quad h \equiv -1 \pmod{m}.$$

Then there exists a good deformation datum with conductor h .

The proof of the proposition is based on the following well-known lemma.

Lemma 1.5 *Let $\omega = f(z) \, dz \in \Omega_{K/k}^1$ be a meromorphic differential form on $X = \mathbb{P}_k^1$. Then ω is logarithmic if and only*

$$\text{ord}_x \omega \geq -1 \quad \text{and} \quad \text{Res}_x \omega \in \mathbb{F}_p,$$

for all $x \in X$.

Proof: Let x_1, \dots, x_r be the set of poles of ω and set $a_i := \text{Res}_{x_i} \omega$. After a change of coordinates we may assume that $x_i \neq \infty$; then the point $x_i \in X$ is defined by $z = z_i$, for some $z_i \in k$.

Now suppose that ω has at most simple poles and that $a_i \in \mathbb{F}_p^\times$, for all i . Choose a lift $A_i \in \mathbb{Z}$ of a_i . Then

$$\omega = \sum_{i=1}^r \frac{a_i \, dz}{z - z_i} = \frac{dg}{g},$$

with

$$g := \prod_{i=1}^r (z - z_i)^{A_i}.$$

This shows one direction of the claimed equivalence. The other direction is obvious. \square

Proof of Proposition 1.4: We fix integers m and h , with $m \geq 1$, $m|(p-1)$, $0 < h < p$ and $h \equiv -1 \pmod{m}$. Write $h = mr - 1$. The condition $h < p$ ensures that there exist elements $z_1, \dots, z_r \in \mathbb{F}_p^\times$ such that

$$z_{i,j} := \zeta_m^j z_i \in \mathbb{F}_p^\times, \quad i = 1, \dots, r, \quad j = 0, \dots, m-1,$$

are pairwise distinct. Define

$$\omega := \frac{dz}{\prod_i (z^m - z_i^m)}.$$

We claim that ω is a good deformation datum of conductor h .

By construction, we have $\sigma^* \omega = \zeta_m \omega$, where σ is the automorphism of X with $\sigma^* z = \zeta_m z$. Furthermore, ω has exactly mr simple poles and no zeroes on $\mathbb{A}_k^1 \subset X$. It follows that ω has a zero of order $h = mr - 1$ at ∞ . Finally, the residues of ω all lie in \mathbb{F}_p . Therefore, ω is logarithmic by Lemma 1.5. This proves the claim and finishes the proof of Proposition 1.4. \square

2 Applications to the lifting problem

Despite its simple proof, Proposition 1.4 has a nontrivial application to the local lifting problem, as already mentioned in the introduction.

First some notation. Let $P = \langle \tau \rangle \cong \mathbb{Z}/p\mathbb{Z}$ be cyclic of order p , with generator τ , and $C = \langle \sigma \rangle \cong \mathbb{Z}/m\mathbb{Z}$ cyclic of order m , with generator σ . Let $\chi : C \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times$ be a character and $G := P \rtimes_\chi C$ the corresponding semi-direct product (such that $\sigma \tau \sigma^{-1} = \tau^{\chi(\sigma)}$, for $\sigma \in C$). Let $\phi : G \hookrightarrow \text{Aut}_k(k[[x]])$ be a local G -action. The *conductor* of ϕ is defined as the positive integer

$$h := \text{ord}_x \left(\frac{\tau(x)}{x} - 1 \right)$$

(compare with (1)). Standard arguments from the theory of local fields give the following restrictions on the conductor h (see [3]). Firstly, h is prime to p . Secondly, the order of h in $\mathbb{Z}/m\mathbb{Z}$ is equal to the order of the character χ (equivalently, $\chi = \chi_0^h$, where $\chi_0 : C \rightarrow K^\times$ is a primitive character of order m). Conversely, if h is a positive integer satisfying these two conditions then there exists a local G -action with conductor h .

However, not every local G -action lifts to characteristic zero. The following theorem gives a necessary and sufficient condition.

Theorem 2.1 *Let $\phi : G \hookrightarrow \text{Aut}_k(k[[x]])$ be a local G -action, where $G = P \rtimes_\chi C$ is the semi-direct product of a cyclic group P of order p and a cyclic group C of order m , with $(m, p) = 1$. Let h be the conductor of ϕ . Then ϕ lifts to characteristic zero if and only if the following two conditions hold.*

- (i) The character $\chi : C \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times$ is either trivial or injective. (Equivalently, the conductor h is either divisible by m or prime to m .)
- (ii) If χ is injective, then the congruence $h \equiv -1 \pmod{m}$ holds.

There are two cases in which Conditions (i) and (ii) are automatically true. The first case is when the character χ is trivial, i.e. when $m|h$. Then G is a cyclic group of order pm , and Theorem 2.1 says that every local G -action lifts to characteristic zero. We recover a result from [6].

The second case is when $m = 2$ and χ is not trivial (or equivalently: h is odd). Then G is the dihedral group of order $2p$. Again, Condition (i) and (ii) are automatically verified, and Theorem 1 says that every local G -action lifts to characteristic zero. This is the main result of [3].

In all other cases (i.e. for $m > 2$ and $m \nmid h$) the congruence $h \equiv -1 \pmod{m}$ is strictly stronger than the condition that h is prime to m (i.e. that χ is injective). This means that not every local G -action lifts to characteristic zero.

Proof of Theorem 2.1: Having available Proposition 1.4, the theorem is a straightforward consequence of the methods of [3]. We only sketch the argument.

By [3], Corollary 3.7, liftability of ϕ is equivalent to the existence of a *Hurwitz tree of type (C, χ) , conductor h and discriminant $\delta = 0$* . Hence the necessity of Conditions (i) and (ii) of Theorem 2.1 follows from [3], Lemma 3.3.(v).

To prove sufficiency of Condition (i) and (ii), it is natural to distinguish the case of small conductor ($h < p$) from the case of large conductor ($h > p$).

Suppose first that $h < p$. By [7], Theorem III.3.1, a Hurwitz tree of conductor $h < p$ is *irreducible*, i.e. it is the same thing as a good deformation datum of conductor h . Therefore, the sufficiency of Conditions (i) and (ii) follows from Proposition 1.4 (and necessity follows directly from Proposition 1.3).

If $h > p$, there may not exist a good deformation datum of conductor h . However, one can easily construct a (not necessarily irreducible) Hurwitz tree of type (C, χ) , conductor h and discriminant $\delta = 0$, by adapting the proof of [3], Theorem 4.1 (which deals with the case $m = 2$ and $h > p$ odd) to the more general situation considered here. This completes the proof of Theorem 2.1. \square

3 The type

In the rest of this paper, we only consider primitive deformation data.

We fix $m \geq 1$ as in Section 1. Let $\omega = f(z)dz$ be a good and primitive deformation datum. After a change of coordinate, we may assume that the unique zero x of ω is located at the point $z = \infty$ (see the proof of Proposition 1.3). This assumption allows us to identify the points different from x with elements of the field k .

Choose a system of representatives z_1, \dots, z_r of the σ -orbits of the set of poles of ω . By Lemma 1.5 we have

$$a_i := \text{Res}_{z_i} \omega \in \mathbb{F}_p^\times.$$

The tuple $\mathbf{a} := (a_1, \dots, a_r)$ is called the *type* of ω .

Condition (a) of Definition 1.1 implies that ω has poles in $z_{i,j} := \zeta_m^j z_i$ and that

$$a_{i,j} := \text{Res}_{z_{i,j}} \omega = \zeta_m^{-jc} a_i, \quad j = 0, \dots, m-1,$$

for some $c \in \mathbb{Z}$, prime to m . It follows that the points $z_{i,j}$ are pairwise distinct. In particular, if $m > 1$ then $z_{i,j} \in k^\times$.

By the residue theorem we have

$$\sum_{i,j} a_{i,j} = \left(\sum_{i=1}^r a_i \right) \cdot \left(\sum_{j=0}^{m-1} \zeta_m^j \right) = 0. \quad (5)$$

If $m > 1$ then the second sum is zero, and the condition (5) coming from the residue theorem is empty.

Definition 3.1 Let $r \geq 2$ be an integer such that $h := mr - 1 \not\equiv 0 \pmod{p}$. A *type* of length r is an r -tuple $\mathbf{a} = (a_1, \dots, a_r)$, with $a_i \in \mathbb{F}_p^\times$. If $m = 1$ then we demand that $\sum_i a_i = 0$. Two types $\mathbf{a} = (a_i)$ and $\mathbf{a}' = (a'_i)$ of length r are equivalent if there exists $\pi \in S_r$ and $c_i \in (\mathbb{Z}/m\mathbb{Z})^\times$ with $a'_i = \zeta_m^{c_i} a_{\pi(i)}$, for $i = 1, \dots, r$.

So the type of a deformation datum ω is well defined, up to equivalence. We can now formulate a refined version of Problem 1.2.

Problem 3.2 Let $\mathbf{a} = (a_1, \dots, a_r)$ be a type. Does there exist a good deformation datum ω of type \mathbf{a} ?

Let $\mathbf{a} = (a_1, \dots, a_r)$ be a type. Choose elements $z_1, \dots, z_r \in k^\times$ such that the elements $z_{i,j} := \zeta_m^j z_i \in k^\times$ are pairwise distinct, for $i = 1, \dots, r$ and $j = 0, \dots, m-1$. Set

$$\omega := \sum_{i=1}^r \frac{m a_i z_i^{m-1} dz}{z^m - z_i^m}.$$

This is clearly a differential form with at most simple poles in the points $z_{i,j}$ satisfying $\sigma^* \omega = \zeta_m \omega$ (Condition (a) of Definition 1.1). A short computation shows that

$$\text{Res}_{z_{i,j}} \omega = \zeta_m^{-j} a_i \in \mathbb{F}_p^\times.$$

By Lemma 1.5, ω is logarithmic (Condition (b) of Definition 1.1). Therefore, ω is a deformation datum of type \mathbf{a} . It is uniquely determined by the choice of the poles z_i .

For which choice of z_1, \dots, z_r is ω a good deformation datum? One can rewrite

$$\omega = \frac{q(z) dz}{\prod_i (z^m - z_i^m)},$$

where $q(z) \in k[z]$ is a polynomial of degree at most $h = mr - 1$ whose coefficients are polynomials in the z_i . The deformation datum ω is good if and only if $q(z)$ is a constant. Setting the nonconstant coefficients of $q(z)$ to zero, we obtain a system of polynomial equations in the unknown z_i whose solutions correspond to good deformation data of type **a**. So far, most attempts to construct good deformation data were based on explicitly solving these equations.

In the remaining sections we suggest another approach to Problem 3.2.

4 A necessary condition

In this section we assume $m = 1$. We formulate a necessary condition on a given type **a** for the existence of good deformation data of type **a** (Corollary 4.2). We then state our main result (Theorem 4.5) which says that this condition is sufficient for large prime p . The proof is given in the following section.

Let ω be a good deformation datum of type **a** = (a_1, \dots, a_r) . We may assume that the point $z = \infty$ is the unique zero of ω . Then

$$\omega = \sum_{i=1}^r \frac{a_i dz}{z - z_i},$$

with $z_1, \dots, z_r \in k$, see the proof of Lemma 1.5. Note that $\sum_i a_i = 0$ and that the conductor of ω is $h = r - 1$, which is prime to p .

For $i = 1, \dots, r$, choose a lift $A_i \in \mathbb{Z}$ of a_i such that

$$\sum_{i=1}^r A_i = 0.$$

We call $\mathbf{A} = (A_1, \dots, A_r)$ a *lift* of the type **a**. Then the rational function

$$g := \prod_{i=1}^r (z - z_i)^{A_i} \tag{6}$$

has the property that $\omega = dg/g$.

We consider g as a morphism $g : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$. Let $I \subset \{1, \dots, r\}$ be the subset of indices i with $A_i > 0$. Note that the degree of g is equal to

$$\sum_{i \in I} A_i = - \sum_{i \notin I} A_i = \sum_{i=1}^r \max(A_i, 0). \tag{7}$$

Proposition 4.1 *The morphism $g : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ is branched exactly above $0, 1, \infty$. More precisely:*

- (i) g is ramified at the point z_i , with ramification index $|A_i|$. Note that $g(z_i) = 0$ if $i \in I$ and $g(z_i) = \infty$ otherwise.
- (ii) g is ramified at ∞ , with ramification index $\geq \min(h, p)$. It is tamely ramified at ∞ if and only if the ramification index is equal to h . Note that $g(\infty) = 1$.
- (iii) g is unramified at all other points.

In particular, if $h < p$ then g is tamely ramified everywhere.

Proof: Claim (i) is clear from the definition of g . Since ω is good and $g(\infty) = 1$, the differential form $dg = g\omega$ has a zero at ∞ of order $h - 1 = r - 2$ and no zero or pole on $\mathbb{A}_k^1 - \{z_1, \dots, z_r\}$. Claim (iii) follows immediately; Claim (ii) as well, but one has to use the fact that the conductor h is prime to p . \square

Given a lifted type $\mathbf{A} = (A_1, \dots, A_r)$, we set

$$n_{\mathbf{A}} := \sum_{i=1}^r \max(A_i, 0), \quad k_{\mathbf{A}} := \gcd(A_1, \dots, A_r). \quad (8)$$

Corollary 4.2 *Let ω be a good deformation datum of type $\mathbf{a} = (a_1, \dots, a_r)$ (with $m = 1$). Then for every lift $\mathbf{A} = (A_1, \dots, A_r)$ of \mathbf{a} we have*

$$k_{\mathbf{A}} \cdot \min(r - 1, p) \leq n_{\mathbf{A}}.$$

Proof: Let $\mathbf{A} = (A_1, \dots, A_r)$ be a lift of \mathbf{a} and $k := k_{\mathbf{A}}$. Then $\mathbf{A}' := (A_1/k, \dots, A_r/k)$ is a lift of $\mathbf{a}' := (a_1/k, \dots, a_r/k)$ with $n_{\mathbf{A}'} = n_{\mathbf{A}}/k$ and $k_{\mathbf{A}'} = 1$. Moreover, $\omega' := k^{-1}\omega$ is a good deformation datum of type \mathbf{a}' . We may therefore assume that $k_{\mathbf{A}} = 1$.

Let $g : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ be the rational function associated to ω and \mathbf{A} by (6). The degree of g is equal to $n_{\mathbf{A}}$ and, by Proposition 4.1 (ii), the ramification index of g at ∞ is $\geq \min(r - 1, p)$. The inequality $\min(r - 1, p) \leq n_{\mathbf{A}}$ follows immediately. \square

Example 4.3 Let $r \geq 4$ be even and $p > r/2$ a prime. Let

$$\mathbf{a} := (\underbrace{1, \dots, 1}_{r/2 \text{ times}}, \underbrace{-1, \dots, -1}_{r/2 \text{ times}}).$$

Then $k_{\mathbf{A}} = 1$ and $n_{\mathbf{A}} = r/2 < \min(p, r - 1)$, for the obvious lift \mathbf{A} . So Corollary 4.2 implies that there does not exist a good deformation datum of type \mathbf{a} .

The special case $p = 5$ and $r = 4$ corresponds to [3], Lemma 4.2. This example had been suggested to us by D. Harbater in connection with the lifting problem.

For $p = 3$ and $r = 6$, there does exist a one-parameter family of good deformation data of type \mathbf{a} . This shows that the bound in Corollary 4.2 is sharp, in some sense.

Example 4.4 Let $s \geq 3$ and $1 \leq \alpha < s$ be integers and $p > s$ be a prime. Set $r := s + 2$ and

$$\mathbf{a} := (\alpha - s, \underbrace{1, \dots, 1}_{s \text{ times}}, -\alpha).$$

Then $k_{\mathbf{A}} = 1$ and $n_{\mathbf{A}} = s < \min(r-1, p)$ for the obvious lift \mathbf{A} . Again, Corollary 4.2 shows that there does not exist a good deformation datum of type \mathbf{a} . This result is proved in [7], Example 4.4, by explicit computations.

Here is our main result.

Theorem 4.5 Let $\mathbf{a} = (a_1, \dots, a_r)$ be a type (with $m = 1$). Suppose that there exists a lift \mathbf{A} of \mathbf{a} such that

$$k_{\mathbf{A}} \cdot (r - 1) \leq n_{\mathbf{A}} < k_{\mathbf{A}} \cdot p.$$

Then there exists a good deformation datum of type \mathbf{a} .

The proof of Theorem 4.5 is given at the end of the following section. It is based on the idea, explained in the introduction, to construct good deformation data from the reduction modulo p of suitable Belyi maps.

5 Good deformation data from Belyi maps

A *Belyi morphism* is a finite morphism $f : Y \rightarrow \mathbb{P}_{\bar{\mathbb{Q}}}^1$, where Y is a smooth projective curve defined over the field $\bar{\mathbb{Q}}$ of algebraic numbers, such that f is branched exactly over the three points 0, 1 and ∞ .

Belyi morphisms can be described by some purely topological data. For instance, one can describe them by drawing a *dessin d'enfants*. An equivalent description, more useful for computations, is the following.

Definition 5.1 Fix an integer $n \geq 1$. A *generating system* of degree n is a triple $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of permutations $\sigma_i \in S_n$ with the following properties:

- $\sigma_1 \sigma_2 \sigma_3 = 1$, and
- $G := \langle \sigma_1, \sigma_2, \sigma_3 \rangle \subset S_n$ acts transitively on $\{1, \dots, n\}$.

Two generating systems $\sigma = (\sigma_i)$ and $\sigma' = (\sigma'_i)$ are called equivalent if there is a permutation $\tau \in S_n$ such that $\sigma'_i = \tau \sigma_i \tau^{-1}$, for $i = 1, 2, 3$.

The *combinatorial type* of a generating system σ is the triple of conjugacy classes of the elements σ_i in S_n :

$$C(\sigma) = (C(\sigma_1), C(\sigma_2), C(\sigma_3)).$$

By a standard construction (see e.g. [9]), we can associate to a generating system σ a Belyi morphism $f : Y \rightarrow \mathbb{P}_{\bar{\mathbb{Q}}}^1$. This gives a one-to-one correspondence between equivalence classes of generating systems of degree n and Belyi morphisms of degree n .

If f is the Belyi morphism corresponding to the generating system σ , then the combinatorial type $C(\sigma)$ can be read off from the ramification structure of f , as follows. Write $(x_1, x_2, x_3) := (0, 1, \infty)$. Let $f^{-1}(x_i) = \{y_1, \dots, y_{s_i}\}$ be the fiber above x_i , and let e_j denote the ramification index of f at the point y_j . Then $C(\sigma_i)$ is the conjugacy class of permutations with cycle structure (e_1, \dots, e_{s_i}) . Note that we can compute the genus of the curve Y solely from the combinatorial type, using the Riemann–Hurwitz formula.

We say that a triple $C = (C_1, C_2, C_3)$ of conjugacy classes of S_n is *realizable* if there exists a generating system σ with $C = C(\sigma)$. This means that there exists a Belyi morphism of degree n with combinatorial type C .

Let $\mathbf{A} = (A_1, \dots, A_r)$ be a lifted type. Let $C(\mathbf{A}) = (C_1, C_2, C_3)$ denote the following triple of conjugacy classes of the symmetric group $S_{n_{\mathbf{A}}}$:

$$C_1 := (A_i \mid i \in I), \quad C_2 := (r-1), \quad C_3 := (-A_i \mid i \notin I) \quad (9)$$

(a conjugacy class in S_n is given as a list of natural numbers indicating the length of the cycles of a permutation, and in which 1s can be omitted). Note that the genus of $C(\mathbf{A})$ is zero.

Lemma 5.2 *The triple of conjugacy classes $C(\mathbf{A})$ is realizable if and only if the inequality*

$$k_{\mathbf{A}} \cdot (r-1) \leq n_{\mathbf{A}}. \quad (10)$$

holds. Here $n_{\mathbf{A}}$ and $k_{\mathbf{A}}$ are defined by (8).

Proof: It is no restriction to assume that

$$A_1, \dots, A_s > 0, \quad A_{s+1}, \dots, A_r < 0, \quad (11)$$

for some $1 \leq s < r$.

Suppose that f is a Belyi map of combinatorial type $C(\mathbf{A})$. We may assume that $f(\infty) = 1$ and that ∞ is the unique ramification point above 1. We consider the inverse image $f^{-1}([-\infty, 0]) \subset \mathbb{C}$ as a *dessins d'enfants*, i.e. a connected bicolored plane graph, as follows: $f^{-1}(0) \subset \mathbb{C}$ is the set of black vertices, $f^{-1}(\infty)$ is the set of white vertices and the connected components of $f^{-1}((-\infty, 0))$ are the edges of the graph. The valency list is (A_1, \dots, A_s) for the black and $(-A_{s+1}, \dots, -A_r)$ for the white vertices. The connected components of $\mathbb{C} - f^{-1}([-\infty, 0])$ are in bijection with the fiber $f^{-1}(1)$. In particular, the (unique) unbounded component corresponds to ∞ .

The fact that f is unramified above 1 except in the point ∞ (where it is ramified of order $r-1$) is equivalent to the following: (*) the boundary of every bounded connected component of $\mathbb{C} - f^{-1}([-\infty, 0])$ consists exactly of one black, one white vertex and two edges.

This condition is equivalent to the condition (*'): if we remove, for all pairs of vertices of different colors, all edges joining the two vertices except for one, we obtain a plane tree. We can thus associate to the Belyi map f a *bicolored weighted plane tree* T_f . Here the weight associated to a edge joining the pair



Figure 1: A dessin of type $(2, 4, -3, -2, -1)$

of vertices (v_1, v_2) is the number of edges joining (v_1, v_2) in the original graph $f^{-1}([-\infty, 0])$.

Conversely, given a bicolored weighted plane tree T (where the weights of the edges are positive integers), there exists a Belyi map f of type \mathbf{A} , unique up to isomorphism, such that T_f is homeomorphic to T , see e.g. [9]. Here the tuple $\mathbf{A} = (A_1, \dots, A_r)$ is the ‘valency list’ of T (black vertices give positive numbers, white vertices give negative numbers), and the valency of a vertex v is the sum of the weights of all edges adjacent to v . See Figure 5 for a tree with valency list $(2, 4, -3, -2, -1)$ and the corresponding dessin.

So far, we have translated the lemma into the following

Claim: There exists a bicolored weighted plane tree T with valency list \mathbf{A} if and only if the inequality (10) holds.

We will only prove the ‘if’ part of the claim and leave the ‘only if’ part as an exercise. So we fix a lifted type $\mathbf{A} = (A_1, \dots, A_r)$ such that (10) and (11) hold and try to construct a tree T with valency list \mathbf{A} (we abbreviate ‘bicolored weighted plane tree’ to ‘tree’).

To begin, we observe that if T is a tree with valency list \mathbf{A} , then the gcd of the valencies, denoted $k_{\mathbf{A}}$ above, also divides the weight of every edge of T (proof by induction on the number of vertices). Multiplying all the weights of a tree by a constant factor has the effect of multiplying all valencies by the same factor. We may therefore assume that $k_{\mathbf{A}} = 1$.

We now prove the claim by induction on the number $\min(s, r - s)$. Suppose first that $\min(s, r - s) = 1$, i.e. $s \in \{1, r - 1\}$. Then one can draw a ‘star shaped’ tree T with valency list \mathbf{A} , as follows. Say $s = 1$. We choose one black vertex $v_1 \in \mathbb{C}$ and $r - 1$ white vertices $v_2, \dots, v_r \in \mathbb{C}$, positioned in counterclockwise order on a circle around v_1 . For $i = 2, \dots, r$ we then join v_1 with v_i by a straight line, which we regard as an edge of weight $-A_i$.

So we may assume that $s, r - s \geq 2$. Then the A_i cannot have all the same absolute value. Therefore, after reordering the A_i (and possibly changing signs) we may also assume

$$A_1 \leq A_2, \dots, A_s, -A_{s+1}, \dots, -A_{r-1}, \quad A_1 < -A_r. \quad (12)$$

Set

$$m_i := \gcd(A_2, \dots, A_i + A_1, \dots, A_r), \quad \text{for } i = s + 1, \dots, r.$$

The assumption $k_{\mathbf{A}} = \gcd(A_1, \dots, A_r) = 1$ implies, by an easy argument, that the numbers m_{s+1}, \dots, m_r are pairwise relatively prime. Moreover, if $A_i = -A_1$ for one index i , then $m_j = 1$ for all $j \neq i$. We conclude

$$\left(\prod_{j=s+1}^r m_j \right) \mid A_i, \quad \text{for } i = 2, \dots, s \quad (13)$$

and

$$\left(\prod_{\substack{j=s+1 \\ j \neq i}}^r m_j \right) \mid A_i, \quad \text{for } i = s+1, \dots, r. \quad (14)$$

Furthermore, we can choose an index $i_0 \in \{s+1, \dots, r\}$ such that

$$m_{i_0} = m := \min(m_{s+1}, \dots, m_r) \quad \text{and} \quad A_{i_0} + A_1 < 0.$$

Set

$$\mathbf{A}' := (A_2, \dots, A_{i_0} + A_1, \dots, A_r) \in \mathbb{Z}^{r-1}.$$

By construction, this is a lifted type with $r' = r - 1 < r$ entries and

$$n_{\mathbf{A}'} = n_{\mathbf{A}} - A_1, \quad k_{\mathbf{A}'} = m$$

(with the obvious notation). Then (13) implies that

$$n_{\mathbf{A}'} = A_2 + \dots + A_s \geq (s-1)m^{r-s}, \quad (15)$$

while (14) implies

$$n_{\mathbf{A}'} = - \left(\sum_{\substack{i=s+1 \\ i \neq i_0}}^r A_i \right) - (A_{i_0} + A_1) \geq (r-s-1)m^{r-s-1}. \quad (16)$$

We claim that

$$k_{\mathbf{A}'} \cdot (r' - 1) \leq n_{\mathbf{A}'}. \quad (17)$$

To prove the inequality (17) we suppose that it is false, i.e. that $m(r-2) > n_{\mathbf{A}'}$. Then (15) implies

$$m^{r-s-1} < \frac{r-2}{s-1} = 1 + \frac{r-s-1}{s-1} \quad (18)$$

and (16) implies

$$m^{r-s-2} < \frac{r-2}{r-s-1} = 1 + \frac{s-1}{r-s-1} \quad (19)$$

Using the assumption $s, r-s \geq 2$, it is easy to deduce from (18) and (19) that $m = 1$. But then the assumed falsity of (17) means that $r-2 > n_{\mathbf{A}} - A_1$, which implies $k_{\mathbf{A}}(r-1) = r-1 > n_{\mathbf{A}}$. This is a contradiction to (10) and shows that (17) is true.¹

¹Although discovered independently, our argument to show (17) is almost identical to the argument given in [10], proof of Theorem 1.

It follows from (17) and the induction hypothesis that there exists a tree T' with valency list \mathbf{A}' . Let v be a white vertex of T' with valency $-(A_{i_0} + A_1)$. Let v' be any point in \mathbb{C} not lying on T' . Let T denote the tree obtained by adding to T' the point v' as a black vertex and an edge with weight A_1 joining v and v' . Then T has valency list \mathbf{A} . This completes the proof of the lemma. \square

Proof of Theorem 4.5: Let $\mathbf{a} = (a_1, \dots, a_r) \in (\mathbb{F}_p^\times)^r$ be a type. Suppose that there exists a lift $\mathbf{A} = (A_1, \dots, A_r)$ such that

$$k_{\mathbf{A}} \cdot (r - 1) \leq n_{\mathbf{A}} < k_{\mathbf{A}} \cdot p.$$

We have to show that there exists a good deformation datum ω of type \mathbf{a} . Replacing \mathbf{a} by $\mathbf{a}' := \mathbf{a}/k_{\mathbf{A}}$, we may assume that $k_{\mathbf{A}} = 1$ (compare with the proof of Corollary 4.2).

By Lemma 5.2, there exists a Belyi map $f : \mathbb{P}_{\bar{\mathbb{Q}}}^1 \rightarrow \mathbb{P}_{\bar{\mathbb{Q}}}^1$ of combinatorial type $C(\mathbf{A})$. After a suitable choice of coordinates, we may assume that $f(\infty) = 1$ and that ∞ is the unique ramification point above 1 (with ramification index $r - 1$). Now we have

$$f = \prod_{i=1}^r (z - z_i)^{A_i},$$

for certain pairwise distinct algebraic numbers $z_1, \dots, z_r \in \bar{\mathbb{Q}}$.

Let v be a place of $\bar{\mathbb{Q}}$ above the prime p . Let k denote the residue field of v . By our hypothesis, the degree of f is $n_{\mathbf{A}} < p$. Therefore, f has *good reduction* at v , by Grothendieck's theory of the tame fundamental group. In our concrete situation, this means that the algebraic numbers z_i are v -integers and that, moreover, the reduction

$$g = \prod_{i=1}^r (z - \bar{z}_i)^{A_i} \in k(z)$$

of f modulo v is at most tamely ramified and branched exactly over $0, 1, \infty$, with ramification structure identical to that of f . It follows that

$$\omega := \frac{dg}{g}$$

is a good deformation datum of type \mathbf{a} . This proves the theorem. \square

6 The case $m = 2$

Now suppose that $m = 2$. Let p be an odd prime and $\mathbf{a} = (a_1, \dots, a_r)$ be a type. A deformation datum of type \mathbf{a} is of the form

$$\omega = \sum_{i=1}^r \left(\frac{a_i dz}{z - z_i} - \frac{a_i dz}{z + z_i} \right) = \sum_{i=1}^r \frac{2a_i z_i dz}{z^2 - z_i^2},$$

where $\pm z_i$ are pairwise distinct elements of k^\times . We suppose that ω is good, i.e. that it has a single zero of order $h := 2r - 1$ at ∞ . Moreover, we assume that $p > h$.

Choose a lift $\mathbf{A} = (A_1, \dots, A_r)$ of \mathbf{a}_i with $A_i > 0$, and set

$$g := \prod_{i=1}^r \left(\frac{z - z_i}{z + z_i} \right)^{A_i}.$$

Then $\omega = dg/g$. Furthermore, we have

$$g(-z) = \prod_{i=1}^r \left(\frac{z + z_i}{z - z_i} \right)^{A_i} = \frac{1}{g(z)}.$$

Therefore, there exists a rational function $\tilde{g} \in k(x)$ such that

$$\left(\frac{g(-z) - 1}{g(-z) + 1} \right)^2 = \left(\frac{g(z) - 1}{g(z) + 1} \right)^2 = \tilde{g}(z^2).$$

Proposition 6.1 *The morphism $\tilde{g} : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ has degree*

$$n = n_{\mathbf{A}} := \sum_{i=1}^r A_i.$$

It is at most tamely ramified and branched exactly above $0, 1, \infty$. More precisely:

- (i) *The fiber $\tilde{g}^{-1}(1)$ consists of the points $x = z_i^2$, for $i = 1, \dots, r$. The ramification index of \tilde{g} at the point $x = z_i^2$ is equal to A_i .*
- (ii) *Suppose that n is even. Then the fiber $\tilde{g}^{-1}(0)$ consists of the point $x = \infty$, with ramification index $h = 2r - 1$, of the point $x = 0$, with ramification index 1, and of $(n - 2r)/2$ points with ramification index 2. The fiber $\tilde{g}^{-1}(\infty)$ consists of $n/2$ points with ramification index 2.*
- (iii) *Suppose that n is odd. Then the fiber $\tilde{g}^{-1}(0)$ consists of the point $x = \infty$, with ramification index $h = 2r - 1$ and of $(n - 2r + 1)/2$ points with ramification index 2. The fiber $\tilde{g}^{-1}(\infty)$ consists of the point $x = 0$, which is unramified, and of $(n - 1)/2$ points with ramification index 2.*

Proof: We have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}_z^1 & \xrightarrow{\psi} & \mathbb{P}_x^1 \\ g \downarrow & & \downarrow \tilde{g} \\ \mathbb{P}_v^1 & \xrightarrow{\phi} & \mathbb{P}_u^1, \end{array}$$

where $\psi(z) = z^2 =: x$ and $\phi(v) = ((v - 1)/(v + 1))^2 =: u$. In particular, $\phi(0) = \phi(\infty) = 1$, $\phi(1) = 0$ and $\phi(-1) = \infty$. So

$$\tilde{g}^{-1}(1) = \psi(g^{-1}(0) \cup g^{-1}(\infty)) = \{z_1^2, \dots, z_r^2\}.$$

Moreover, since g is ramified at $z = \pm z_i$ of order A_i (Lemma 1.5) and ψ is unramified at $z = \pm z_i$, we see that the ramification index of \tilde{g}^{-1} at $x = z_i^2$ is equal to A_i . This proves (i).

One proves (ii) and (iii) in the same manner. The case distinction comes from

$$g(0) = (-1)^n \quad \Rightarrow \quad \tilde{g}(0) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \infty, & \text{if } n \text{ is odd.} \end{cases}$$

□

For a lift $\mathbf{A} = (A_1, \dots, A_r)$ of \mathbf{a} we set $C(\mathbf{A}) = (C_1, C_2, C_3)$, where

$$C_1 := (2r-1, \underbrace{2, \dots, 2}_{[(n-2r+1)/2]}), \quad C_2 := (A_1, \dots, A_r), \quad C_3 := (\underbrace{2, \dots, 2}_{[n/2]})$$

are conjugacy classes in $S_{n_{\mathbf{A}}}$. With this notation, we have obtain the following version of Corollary 4.2 and Theorem 4.5 for $m = 2$.

Theorem 6.2 *Let \mathbf{a} be a type, with $m = 2$ and $p > h = 2r - 1$.*

- (i) *If there exists a good deformation datum ω of type \mathbf{a} , then the triple of conjugacy classes $C(\mathbf{A})$ is realizable, for every lift \mathbf{A} of \mathbf{a} .*
- (ii) *Suppose that there exists a lift \mathbf{A} of \mathbf{a} such that*
 - (a) *$C(\mathbf{A})$ is realizable, and*
 - (b) *$n_{\mathbf{A}} < p$.*

Then there exists a good deformation datum ω of type \mathbf{a} .

We have not tried to find an explicit criterion for realizability in the above situation (such as Lemma 5.2). Instead, we give an example:

Example 6.3 For an arbitrary $r \geq 1$, set $m := 2$ and

$$\mathbf{A} := (\underbrace{1, \dots, 1}_{r-1}, r).$$

Then $n = h = 2r - 1$ and the triple $C(\mathbf{A})$ consists of the following conjugacy classes in S_n : C_1 is the class of all n -cycles, C_2 the class of r -cycles and C_3 the class of the product of $r - 1$ disjoint 2-cycles.

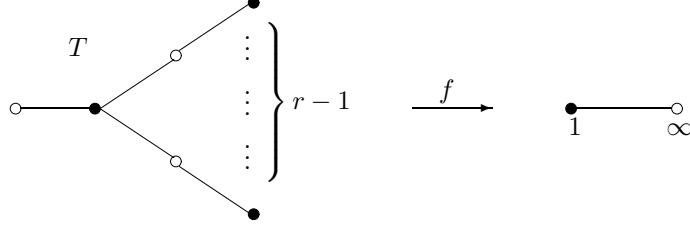
Lemma 6.4 *Up to equivalence, there is a unique generating system σ of combinatorial type $C(\mathbf{A})$.*

Proof: The existence of σ follows from the identity

$$(1 \ 2 \ \dots \ n) \cdot (n \ n-2 \ \dots \ 3 \ 1) = (2 \ 3)(4 \ 5) \cdots (n-1 \ n).$$

The proof of the uniqueness is an easy exercise. □

One can also convince oneself of the truth of Lemma 6.4 by drawing the corresponding dessin d'enfant. If $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a Belyi map of combinatorial type $C(\mathbf{A})$, with \mathbf{A} as above, then the inverse image $T := f^{-1}([1, \infty]) \subset \mathbb{C}$ is a *pre-clean plane tree* with valency list $(h, 1, \dots, 1)$ (with r repetitions of 1), see e.g. [9]. It is clear that there is, up to isomorphism, a unique realization of such a tree:



Corollary 6.5 *Let $r \geq 1$ and $p > n := 2r - 1$ a prime number. Set $m := 2$. Then there exists a unique good deformation datum of type $\mathbf{a} := (1, \dots, 1, r)$, defined over \mathbb{F}_p .*

This is essentially what is proved in [3], Section 5.3, by showing that a certain system of equations has a unique solution. The above proof is clearly shorter and more elegant.

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